# PRIMAL-DUAL ALGORITHMS

## Hitting set

<u>in:</u> • ground set $\mathcal{U} = [n]$ w/ costs $c_i \geq 0$ for $i \in [n]$

• set system $\mathcal{S} = \{S_1, \ldots, S_m\}$, $S_i \subseteq \mathcal{U}$

<u>goal:</u> • a subset $U \subseteq \mathcal{U}$ s.t. $\forall\, S_i \in \mathcal{S}$ we have

$U \cap S_i \neq \emptyset$, i.e. every set is hit by some picked element

• minimize the cost $\sum_{i \in U} c_i$

LP relaxation for Hitting Set

$$\min \sum_{i=1}^{n} c_i x_i \qquad // \; x_i = \begin{cases} 1 & \text{if } i \text{ is in the sol} \\ 0 & \text{o/w} \end{cases}$$

$$\sum_{x_j \in S_i} x_j \geq 1 \quad \text{for } i \in [m] \qquad \text{// every set has to have} \geq 1 \text{ elem picked}$$

$$x_j \geq 0 \quad \text{for } j \in [n]$$

Dual

$$\max \sum_{i=1}^{m} y_i$$

$$\sum_{j: i \in S_j} y_j \le c_i \qquad \text{for } i \in [n]$$

$$y_i \ge 0 \qquad \text{for } i \in [m]$$

$$A = \begin{array}{c} \begin{matrix} x_1 & \dots & x_i & \dots & x_n \end{matrix} \\ \left( \begin{matrix} & & & & \\ & & & & \\ & & & & \end{matrix} \right) \end{array}$$

when does an entry of this column have entry 1?

$j$th row has 1 if $i \in S_j$

Why do we care? So we can interpret the dual.

Alg:

1. $\vec{y} = \vec{0}$ // feasible solution of (D) ($\Leftarrow c_i \ge 0$)
2. $U \leftarrow \emptyset$ // "infeasible sol of (P)"
3. while $\exists i \in [m]$ s.t. $S_i \cap U = \emptyset$
4. increase $y_i$ until a constraint of (D) becomes tight
   - let $l$ be that constraint $\Rightarrow \sum_{i: l \in S_i} y_i = c_l$
5. $U \leftarrow U \cup \{l\}$

- we will see that this is an $f$-apx alg where

$$f = \max_{i \in [m]} |S_i|$$

- we want to show $ALG = \sum_{i \in U} w_i \leq f \cdot OPT$
- as always, we want a lower bound on OPT s.t. we know how to compare it w/ ALG

  best hitting set

- let $Z^*_{LP}$ be the opt of (P) $\Rightarrow$ $Z^*_{LP} \leq OPT$ since LP relaxation blah blah
- what is $\vec{y}$? A feasible sol. of (D)

$\rightarrow$ $\sum_{i=1}^{m} y_i \leq Z^*_{LP}$ by weak duality

value of $\vec{y}$ as given by the obj. fct of (D)

- thus one such lower bound would be $\sum_{i=1}^{m} y_i$ and the goal is to show

$$\sum_{i \in U} w_i \leq f \cdot \sum_{j=1}^{m} y_j$$

- when is an elem $e \in \mathcal{U}$ in the sol $U$?
- by Alg, when the corresponding constraint is tight
- so $\sum_{i \in U} w_i = \sum_{i \in U} \sum_{j : i \in S_j} y_j$

- now we swap the sums

$$\sum_{i\in U} w_i = \sum_{j=1}^{m} y_j \, |\{i \in U : i \in S_j\}|$$

- what's this ? Who knows, but it's at most the size of the biggest set

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- is this a good bound?
- let's model vertex cover as a hitting set
  - elements are vertices
  - we have a set for each $\{u,v\} \in E$
  - here our $f = 2$ ... pretty good

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<u>Primal-dual algorithm template</u>

1. obtain (P) and (D)
2. initialize an empty sol. of the problem and a trivial feasible sol of the dual
3. increase some variables of (D) until a constraint becomes tight and add the corresponding input elem into the sol
4. repeat 3. until sol of (P) becomes feasible

Analysis rationale

- suppose $x^*$ is an opt IP 0/1 sol (of Hitting Set, VC, ...)
- from the previous tutorial, recall complementary slackness conditions
- if vertex/elem/... $u$ is in the opt sol, $x^*_u = 1$
- $x^*_u > 0 \Rightarrow$ corresponding constraint in (D) is tight by CSC
- we work in "reverse", i.e. we make dual variables nonzero
- if $y_j > 0$ in our sol implied tightness in the primal, then CSC would imply that our primal sol would be optimal
- that's unfortunately not true, but what we usually have is

$$y_i > 0 \Rightarrow \sum_{j:\, j \text{ "corresponds to } i\text{"}} x_j \leq f$$

- and as $x^*_i \in \{0,1\}$, $f$ is an $f$-apx

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Shortest path alg.

- recall the P-D alg from the lecture
- we want to show that edges added to the sol are added in the same manner as if by Dijkstra's alg

- we increase $y_S$ in every step until there exists an edge in $\partial(S)$ s.t. its corresponding $\sum_{S: e \in \partial(S)} y_S \le c_e$ is violated

and $e$ is added to the sol

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- Dijkstra maintains a tree $T$ s.t. any path from $u$ to a vertex $u \in T$ in $T$ is in fact the shortest $s-u$ path
- P-D alg maintains a tree too ... is it the same one?
- how to prove it?
- redo the analysis but replace $t$ by $u$ :)