What is linear programming (LP)?

- the goal is to find a vector $\vec{x} \in \mathbb{Q}^n$ which minimizes some linear fct under some linear constraint
- we will formally specify an LP task for $\vec{x} \in \mathbb{Q}^n$, $\vec{b} \in \mathbb{Q}^m$, $A \in \mathbb{Q}^{m \times n}$, $\vec{c} \in \mathbb{Q}^n$ as follows

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} A_{ij} x_j \geq b_i \ , \quad i \in \{1, \dots, m\}$$

$$x_j \geq 0 \ , \quad j \in \{1, \dots, n\}$$

- or in vector/matrix form

$$\min \ c^T x$$

$$\text{subject to} \quad A\vec{x} \geq \vec{b}$$

$$\vec{x} \geq \vec{0} \quad \text{coordinate-wise}$$

- a feasible solution is any vector $\vec{x} \in \mathbb{Q}^n$ that satisfies the constraints

$$A\vec{x} \geq \vec{b} \ , \quad \vec{x} \geq \vec{0}$$

- a solution to an LP is a feasible solution which minimizes the obj. fct $c^T x$
- if an LP has a feasible sol., then we say that the LP is feasible, o/w it's infeasible

- there are many other forms of LP's
  - we maximize instead of minimizing
  - $Ax = b$
  - ...
- however, the form above is sufficient for our purposes
- thus we call it the canonical form
- it should be easy to see that we can go between various forms of LPs as needed

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Integer programming (IP)

- require that $x \in \mathbb{Z}^n$ instead of $\mathbb{Q}^n$
- $\underbrace{\text{unlike linear programming}}$, IP is NP-hard
  - ellipsoid method, interior point method, ... are poly algs

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Duality

- example

$$\begin{aligned} \min \quad & 6x_1 + 4x_2 + 2x_3 \\ \text{subj. to} \quad & 4x_1 + 2x_2 + x_3 \geq 5 \\ & x_1 + x_2 \geq 3 \\ & x_2 + x_3 \geq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- we are looking for a lower bound on the opt value
- since x_i's ≥ 0 we can compare obj. fct and the first constraint

$$6x_1 + 4x_2 + 2x_3 \geq 4x_1 + 2x_2 + x_3 \geq 5$$

$\rightarrow$ $OPT \geq 5$

- we can take the first constraint and twice the second constraint

$$6x_1 + 4x_2 + 2x_3 \geq 4x_1 + 2x_2 + x_3 + 2(x_1+x_2) \geq 5 \cdot 2 \cdot 3 \geq 11$$

- similarly, adding all three constraints, we get $OPT \geq 12$

- wait, we are trying to optimize a linear fct under linear constraints? Let's get an LP to do that

- we want

constraints

$$6x_1 + 4x_2 + 2x_3 \geq y_1(4x_1 + 2x_2 + x_3) + y_2(x_1+x_2) + y_3(x_2+x_3)$$

- and since the x_1's on RHS should add up to ≤ 6, then

$$4y_1 + 1 \cdot y_2 \leq 6 \qquad \text{for } x_1$$

- and similarly for x_2, x_3

$$2y_1 + y_2 + y_3 \leq 4$$

$$y_1 + y_3 \leq 2$$

$$y_i \geq 0$$

- together, we have a new LP

$$\begin{aligned} \max \quad & 5y_1 + 3y_2 + 2y_3 \\ \text{subj. to} \quad & 4y_1 + y_2 \le 6 \\ & 2y_1 + y_2 + y_3 \le 4 \\ & y_1 + y_3 \le 2 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

- this is called a *dual* of the original LP
- for each constraint of the program, we introduce a variable

- recipe for creating duals

(P) ... primal

$\vec{c}, \vec{x} \in \mathbb{Q}^n$, $\vec{b} \in \mathbb{Q}^m$, $A \in \mathbb{Q}^{m \times n}$

$$\begin{aligned} \min \quad & \vec{c}^T \vec{x} \\ \text{subj. to} \quad & A\vec{x} \ge \vec{b} \\ & \vec{x} \ge \vec{0} \end{aligned}$$

(D) ... dual

$y \in \mathbb{Q}^m$

$$\begin{aligned} \max \quad & \vec{b}^T \vec{y} \\ \text{subj. to} \quad & A^T \vec{y} \le \vec{c} \end{aligned}$$

<u>Thm</u> (Weak duality) If $\vec{x}$ is a feasible solution to (P) and $\vec{y}$ is a feasible sol. to (D), then $\vec{c}^T\vec{x} \ge \vec{b}^T\vec{y}$.

<u>Pf</u>:

- $\vec{c}^T\vec{x} = \sum_{i=1}^{n} c_i x_i$
- from the constraints of the dual $\quad \sum_{i=1}^{m} a_{ij} y_i \le c_j$ for $j \in [m]$

we have

$$\sum_{i=1}^{n} c_i x_i \ge \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ji} y_j \right) x_i$$

• swap the sums

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ji} y_j\right) x_i = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} x_i\right) y_j$$

• from the constraints $\sum a_{ji} x_i \geq b_j$ we have

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji} x_i\right) y_j \geq \sum_{j=1}^{m} b_j y_j \qquad \square$$

• there's also strong duality, i.e. opt of (P) = opt of (D).

• but we won't prove it

Complementary slackness conditions

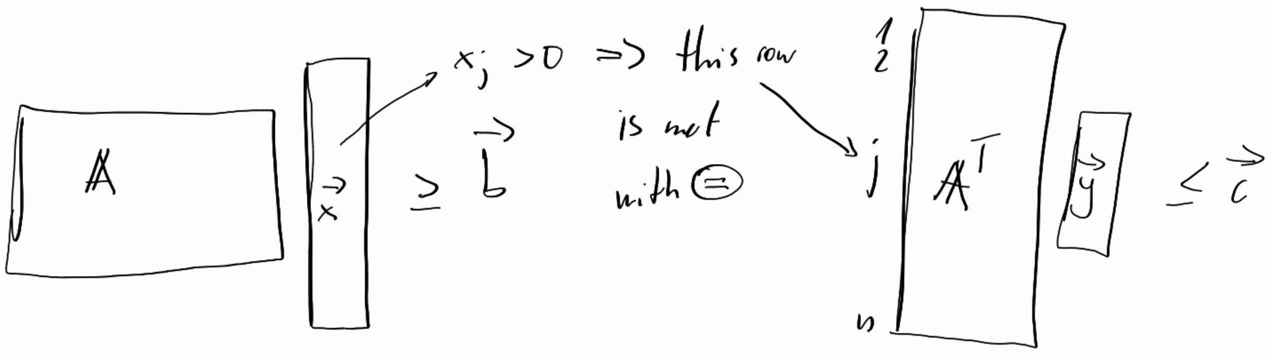
• let $\vec{x}$ and $\vec{y}$ be sols of (P) resp. (D)

• $\vec{x}$ and $\vec{y}$ obey <u>complementary slackness conditions</u> if

$$\sum_{i=1}^{m} a_{ij} y_i = c_j \quad \text{whenever} \quad x_j > 0, \quad \text{and}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{whenever} \quad y_j > 0$$

• i.e. if $x_j > 0$, then the dual constraint corresponding to x_j is met w/ equality and similarly for y_i



• as a crly of duality, we have the following

<u>Crly</u> (Complementary slackness) Let $\vec{x}$ and $\vec{y}$ be feasible sols of (P) resp. (D). Then $\vec{x}$ and $\vec{y}$ obey complementary slackness conditions iff $\vec{x}$ and $\vec{y}$ are opt for their LPs.

<u>Pf</u>: (<=)

• $\vec{x}$ and $\vec{y}$ are opt? By strong duality we have

$$\sum_{i=1}^{n} c_i x_i = \sum_{j=1}^{m} b_j y_j$$

• now inspect the proof of weak duality

• and by strong duality ↗, so the middle is equal too

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ji} y_j \right) x_i \boxed{=} \sum_{j=1}^{n} \boxed{b_j} y_j$$

$$= \sum_{j=1}^{m} \boxed{\left(\sum_{i=1}^{n} a_{ji} x_i \right)} y_j \qquad \boxed{=}$$

equality? Yes ☺

• now =>

• complementary slackness conditions are obeyed

Thm: For a primal (P) and its dual (D), one of the following must be true

(i) (P) and (D) are feasible

(ii) (P) is infeasible and (D) is unbounded

(iii) (P) is unbounded and (D) is infeasible

(iv) both (P) and (D) are infeasible

Formulate WEIGHTED VERTEX COVER as an IP (weights are positive)

- for each vertex v we introduce a variable x_v
- for each edge $uv \in E$, add a constraint

$$x_u + x_v \geq 1$$

- each vertex is either in the solution or not

$\rightarrow x_u \in \{0,1\} \quad \forall u \in V$

- we want a sol of min size

$$\min \sum_{i=1}^{n} w_i x_i$$

- if we relax $x_u \in \{0,1\}$ $\forall u \in V$ to $x_u \geq 0$, then we get an

LP relaxation of the orig. IP

- let's see how to get a 2-apx alg from the opt sol of the LP relaxation
- let OPT_{LP} and OPT_{IP} be opt sols of the LP relaxation an IP respectively
- note that $OPT_{IP} = OPT_{(VC)}$
- observe that $OPT_{LP} \leq OPT_{IP}$ ... why?
  - OPT_{IP} is a feasible sol of LP but not necessarily the other way around

- let's inspect a constraint

$x_u + x_v \geq 1$

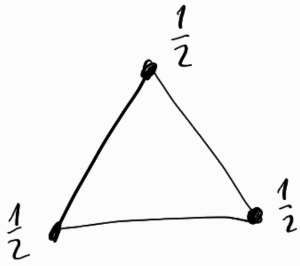
- then x_u or x_v is at least $\frac{1}{2}$
- we create a WVC as follows
  - if $x_u \geq \frac{1}{2}$, then we set x_u to 1 and add u to the vertex cover
  - if $x_u < 0$, we set x_u to 0
- is this a feasible solution? i.e. is each constraint of LP still satisfied?
- yes. every constraint had to have at least one var of $x_u + x_v \geq 1$ at value $\geq \frac{1}{2}$ $\Rightarrow$ at least one of them got set to 1
- what is the cost of our solution?

$\sum w_i x_i$?

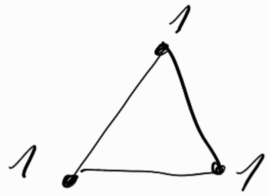
- suppose $w_j x_j > 0 \Rightarrow x_j > 0$
- if $x_j \geq \frac{1}{2}$, we increased it to 1
- let x_j' be rounded x_j
- then $\sum_{j=1}^{n} w_j x_j' \leq 2 \sum_{j=1}^{n} w_j x_j = 2\, OPT_{LP} \leq 2 OPT_{IP}$ $\square$

$\rightarrow$ 2-apx

- can this alg be improved?



$OPT_{LP} = 1{,}5$



rounded sol $= 3$:(