1. The goal of this task is to prove that approximating metrics deterministically is not really possible.

   Let $C_n = (V, E)$ be a cycle on n vertices and $d_{u,v}$ be the distance between $u, v \in V$ on C_n. Show that for any tree metric (V, T) **on the same set of vertices**[1] V, there must exist a pair of vertices $u, v \in V$ such that $d_{u,v} = 1$, but $T_{u,v} \geq \Omega(n-1)$.

   To do this, suppose that of all trees T with optimal distortion α, T has minimum total length. What does T look like?

2. In the UNIVERSAL TSP problem, we are given as input a metric space (V, d) and we must construct a tour π of the vertices. Let π_S be the tour of the vertices $S \subseteq V$ given by visiting them in the order given by the tour π. Let OPT_S be the value of an optimal tour on the metric space induced by the vertices $S \subseteq V$. The goal of the problem is to find a tour π that minimizes $\frac{\pi_S}{\text{OPT}_S}$ over all $S \subseteq V$. In other words, we would like to find a tour such that for any subset $S \subseteq V$, visiting the vertices of S in the order given by the tour is close in value to the optimal tour of S.

   Show that if (V, d) is a tree metric, then it is possible to find a tour π such that $\pi_S = \text{OPT}_S$ for all $S \subseteq V$.

3. In the k-MEDIAN problem, we are given a metric (V, dist) and a parameter k. We wish to select a subset $S \subseteq V$ of k centers to minimize the sum of the distances of each location to its nearest center. That is, we wish to select $S \subseteq V$ with $|S| = k$ to minimize $\sum_{j \in V} \min_{i \in S} \text{dist}(i, j)$.

   - Give a polynomial-time algorithm in the case that the metric dist comes from a tree metric (V, T). You can assume that you know the tree T and that T is (w.l.o.g.) a binary tree.
   - Using the algorithm in the preceding part, give a randomized $\mathcal{O}(\log n)$-approximation algorithm for k-MEDIAN.

[1] This is quite a restriction. However, it is known that even if $V(T) \setminus V \neq \emptyset$, the same result would still hold (albeit with a different multiplicative constant).