Capacitated k-Center in Low Doubling and Highway Dimension

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Capacitated k-Center

Input

- graph $G = (V, E)$ with edge lengths $\ell\colon E \to \mathbb{R}^+$,
- integer k,
- *capacities* $c\colon V \to \mathbb{N}$.

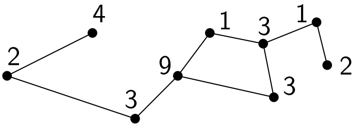
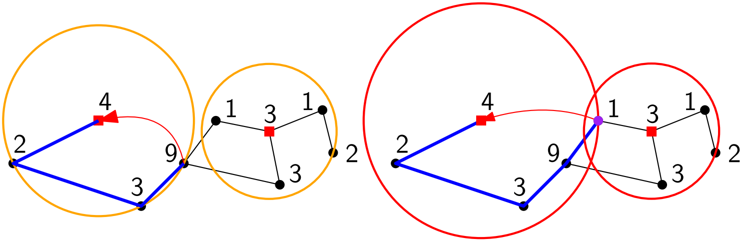


Figure: CKC input with $k = 2$.

Capacitated k-Center: Goal

Find $S \subseteq V$ and an *assignment* $\varphi\colon (V \setminus S) \to S$ such that

- $|S| \leq k$,
- for every $u \in S$, $|\varphi^{-1}(u)| \leq c(u)$, and
- $\max_{v \in V \setminus S} \operatorname{dist}(v, \varphi(v))$ is minimal.



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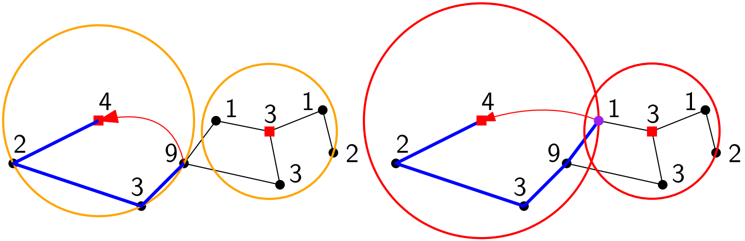

When $c(u) = |V|$ for every $u \in V$ $\Rightarrow$ k-Center.

Capacitated k-Center: Solution Prospects

CAPACITATED k-CENTER is NP-hard.

$\Rightarrow$ cannot solve exactly in polynomial time assuming $\mathrm{P} \neq \mathrm{NP}$.

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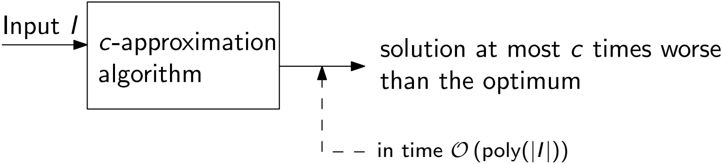
c-approximation algorithm



Capacitated k-Center: Solution Prospects

CAPACITATED k-CENTER is NP-hard.

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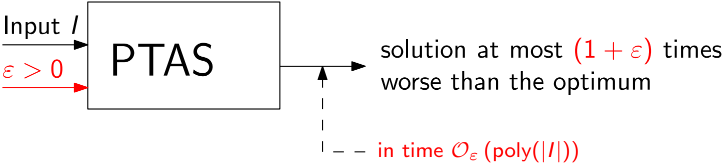
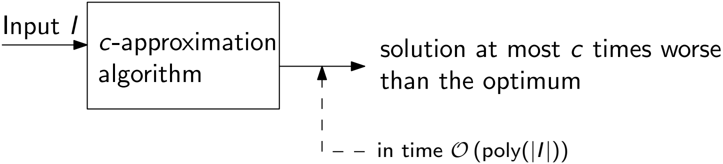
Polynomial-time approximation scheme



Capacitated k-Center: Solution Prospects

c-approximation algorithm



An, Bhaskara, Chekuri, Gupta, Madan, Svensson. 2015

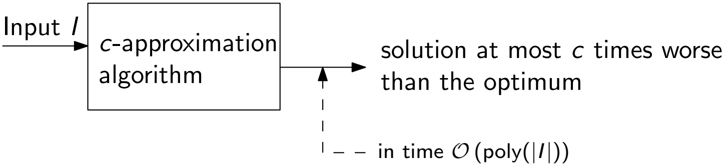
There is a 9-approximation algorithm for CKC.

Cygan, Hajiaghayi, Khuller. 2012

There is no $(3-\varepsilon)$-approximation algorithm for CKC unless P $=$ NP.

Capacitated k-Center: Solution Prospects

c-approximation algorithm



Cygan, Hajiaghayi, Khuller. 2012

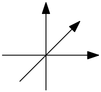
There is no $(3-\varepsilon)$-approximation algorithm for $\textsc{CkC}$ unless P $=$ NP.

Question

Are there settings where we can overcome this lower bound?

Planar graphs, Euclidean spaces, real world, ...

Special Settings?



	Doubling Dimension (Δ)	
Capacitated k-Center	generalizes the dimension of ℓ_q spaces	
k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Feldmann, Marx. 2020	
k-Median, k-Means, Facility Location	$2^{(1/\varepsilon)^{\mathcal{O}(\Delta^2)}} \cdot \text{poly}(n)$ Cohen-Addad, Feldmann, Saulpic. 2021	
TSP, Steiner Tree	$\exp\{2^{\mathcal{O}(\Delta)} \cdot (4\Delta \log n/\varepsilon)^{\Delta}\}$ Talwar. 2004	

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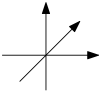

	Doubling Dimension (Δ)	Highway dimension (h)
Capacitated k-Center		captures properties of transportation networks
k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \mathrm{poly}(n)$ Feldmann, Marx. 2020	$f(k,h,\varepsilon) \cdot \mathrm{poly}(n)^\dagger$ Becker, Klein, Saulpic. 2018
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†: f: computable function

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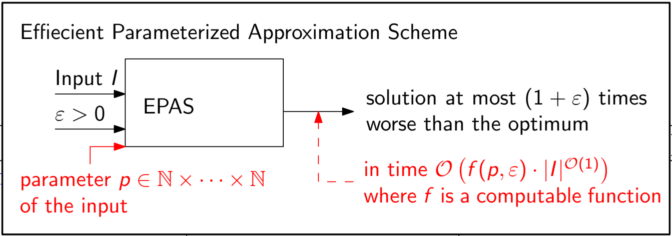

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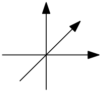

	Doubling Dimension (Δ)	Highway dimension (h)
CAPACITATED k-CENTER	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \mathrm{poly}(n)$ Theorem 2	
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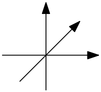

	Doubling Dimension (Δ)	Highway dimension (h)
CAPACITATED k-CENTER	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Theorem 2	$\exists c > 1$: no c-approximation in $\mathcal{O}_\varepsilon\left(f(k,h) \cdot \text{poly}(n)\right)^{\dagger,\S}$ Theorem 1
k-CENTER	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Feldmann, Marx. 2020	$f(k,h,\varepsilon) \cdot \text{poly}(n)^{\dagger}$ Becker, Klein, Saulpic. 2018
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Doubling Dimension

- Let $M = (X, \text{dist})$ be a metric space.

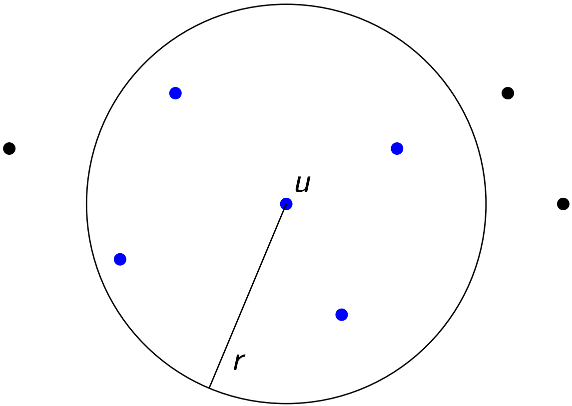
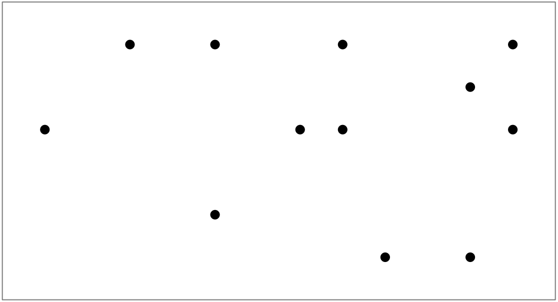


Figure: $B_r(u)$: Ball of radius r.

Doubling Dimension

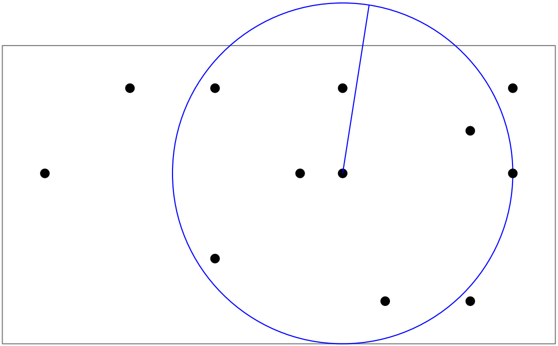
Doubling dimension $\Delta(M)$: smallest $\Delta \in \mathbb{N}$ such that



Doubling Dimension

Doubling dimension $\Delta(M)$: smallest $\Delta \in \mathbb{N}$ such that

- the ball $B_r(u)$ for every $u \in X$ and every $r \in \mathbb{R}^+$



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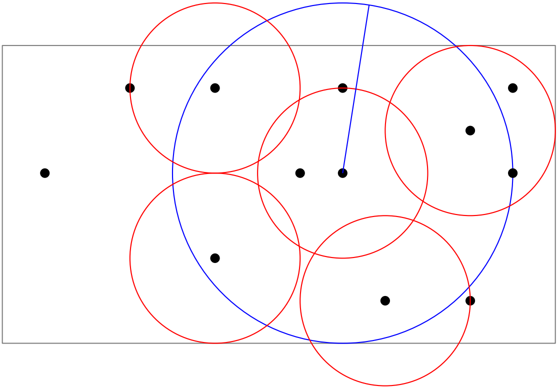
- the ball $B_r(u)$ for every $u \in X$ and every $r \in \mathbb{R}^+$
- is contained in $\cup_{v \in V} B_{r/2}(v)$ for some $V \subseteq X$ with $|V| \leq 2^\Delta$.



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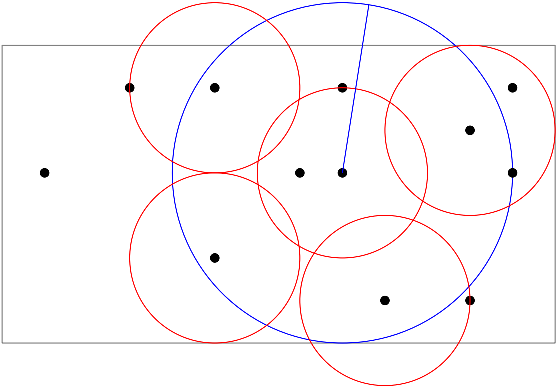
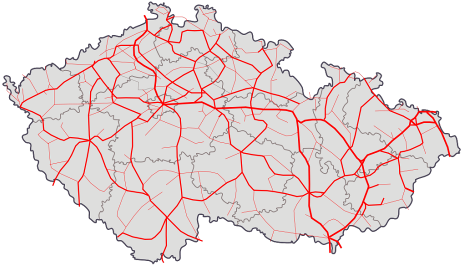


$\rightsquigarrow$ d-dimensional ℓ_q metrics have doubling dimension $\mathcal{O}(d)$.

Highway Dimension: Shortest Path Cover

- Let G be an edge-weighted graph and fix a *scale* $r \in \mathbb{R}^+$.
- Let $\mathcal{P}_r$ be the set of paths of G such that
  - they are a shortest path between their endpoints,
  - their length is more than r and at most $2r$.

(a) Metro and tram network in Prague city center.



(b) Czech railway network.

Highway Dimension: Shortest Path Cover

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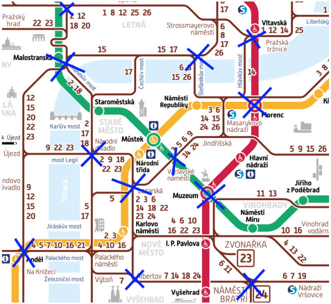
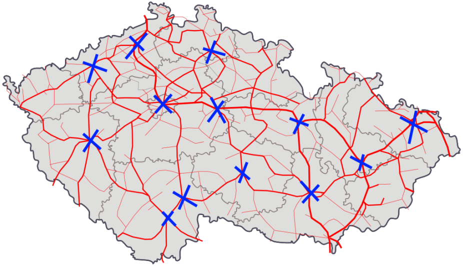




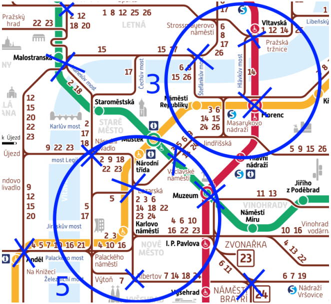
The *shortest path cover* $\mathrm{SPC}_r(G)$ is a hitting set[1] for $\mathcal{P}_r$.

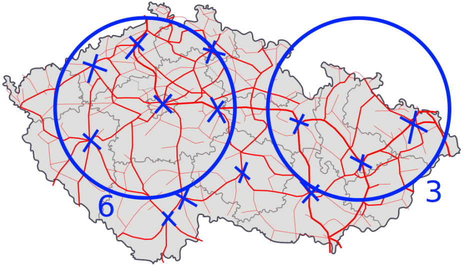
[1] For every $P \in \mathcal{P}_r$ we have $P \cap \mathrm{SPC}_r(G) \neq \emptyset$.

Highway Dimension

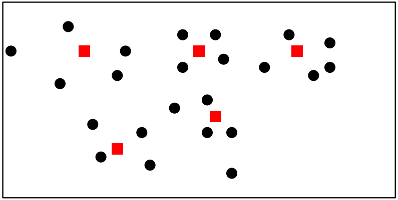
highway dimension of an edge-weighted graph G:

- smallest integer h such that,
- for any scale $r \in \mathbb{R}^+$,
- there exists $H := \mathrm{SPC}_r(G)$ so that,
- $|H \cap B_{2r}(u)| \leq$ h for every $u \in V(G)$.



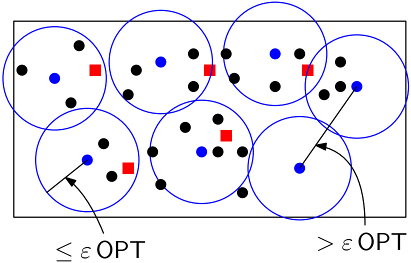


k-CENTER algorithm



■ Optimum solution of cost OPT.

k-Center algorithm

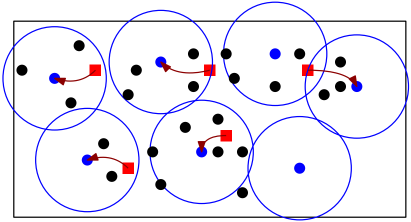


Optimum solution of cost OPT.

Net: $Y \subseteq X$ such that

- $\forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon\,\mathrm{OPT}$, and
- $\forall y_1 \neq y_2 \in Y : d(y_1, y_2) > \varepsilon\,\mathrm{OPT}$.

k-Center algorithm



- Optimum solution of cost OPT.
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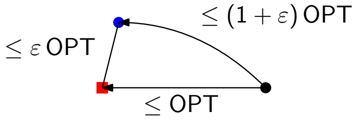
  - $\forall y_1 \neq y_2 \in Y: d(y_1, y_2) > \varepsilon\,\mathrm{OPT}$.

Replace every optimum center by its nearest net point.

$\Rightarrow$ We get a $(1+\varepsilon)$-approximate solution.



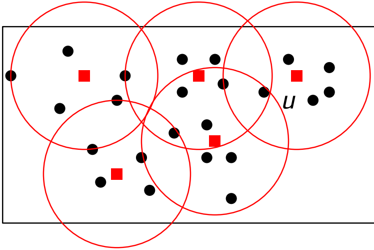
k-Center algorithm

guess OPT
for all $K \subseteq Y$ with $|K| \leq k$ **do**
    if $\bigcup_{u \in K} B_{(1+\varepsilon)\,\mathrm{OPT}}(u) \supseteq V$ **then**
        return K
return :(

Bounding net size

Gupta, Krauthgamer, Lee. 2003

Let M be a metric and $\alpha(M) = \frac{\max_{u,v \in M} \text{dist}(u,v)}{\min_{u \neq v \in M} \text{dist}(u,v)}$. Then for every $M' \subseteq M$ we have $\Delta(M') \leq \Delta(M)$ and $|M| \leq 2^{\mathcal{O}(\Delta \lceil \log_2(\alpha) \rceil)}$.



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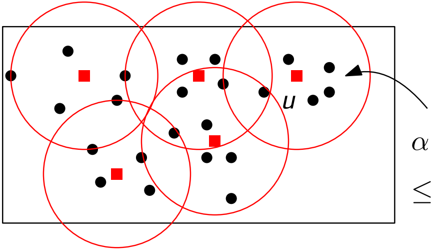

$\alpha = \frac{2\,\text{OPT}}{\varepsilon\,\text{OPT}} = 2\varepsilon^{-1}$

$\leq \varepsilon^{-\mathcal{O}(\Delta)}$ points in $B_{\text{OPT}}(u)$

Bounding net size

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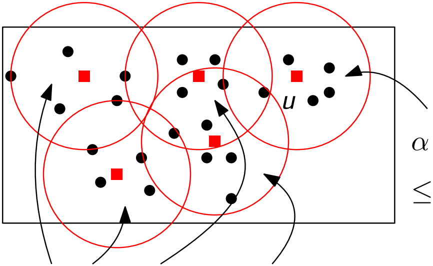

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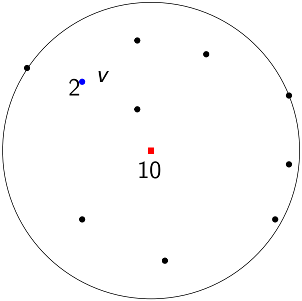
$\leq \varepsilon^{-\mathcal{O}(\Delta)}$ points in $B_{\text{OPT}}(u)$

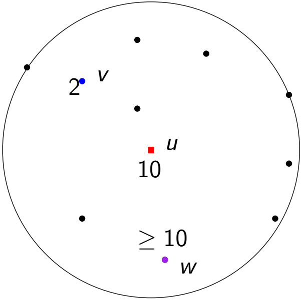
k-Center? $|Y| \leq k\varepsilon^{-\mathcal{O}(\Delta)}$

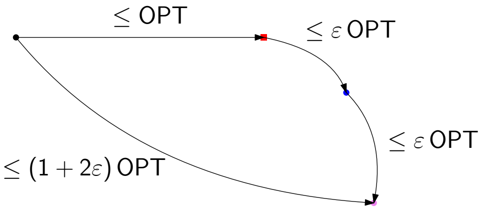
CKC algorithm obstacles



- $c(u) < c(v)$?

CKC algorithm obstacles



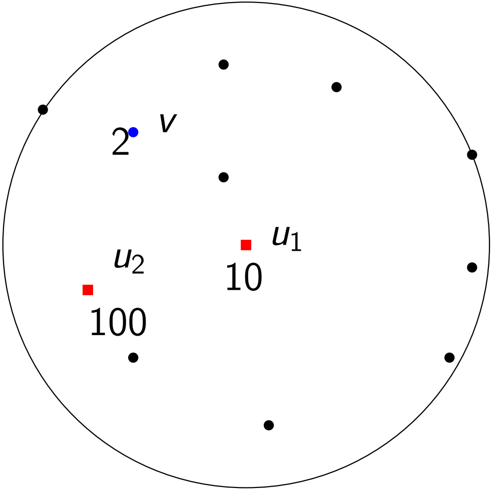


- $c(u) < c(v)$?
- w: vertex with highest capacity in $B_{\varepsilon \text{OPT}}(u)$.
- $c(w) \geq c(v)$ as v, the optimum center, exists.

CKC algorithm

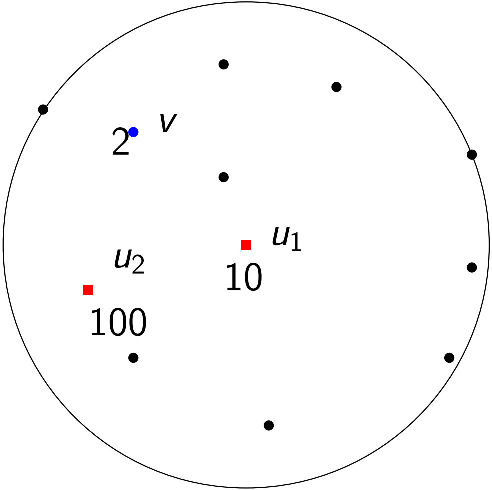
guess OPT
for all $K \subseteq Y$ with $|K| \leq k$ **do**
    $S \leftarrow \emptyset$
    for all $v \in K$ **do**
        $w \leftarrow$ vertex with the highest capacity in $B_{\varepsilon \mathrm{OPT}}(v)$
        $S \leftarrow S \cup \{w\}$
    if ??? **then**
        return S
return :(

CkC algorithm obstacles



- multiple optimum centers in $B_{\varepsilon\,\mathrm{OPT}}(v)$?

CKC algorithm obstacles



- multiple optimum centers in $B_{\varepsilon\,\mathrm{OPT}}(v)$?
- use multisets!

CKC algorithm

guess OPT

for all $K \subseteq_{\text{multiset}} Y$ with $|K| \leq k$ **do**

  $S \leftarrow \emptyset$

  for all $v \in K$ **do**

    $w \leftarrow$ vertex with the highest capacity in $B_{\varepsilon\,\text{OPT}}(v) \setminus S$

    $S \leftarrow S \cup \{w\}$

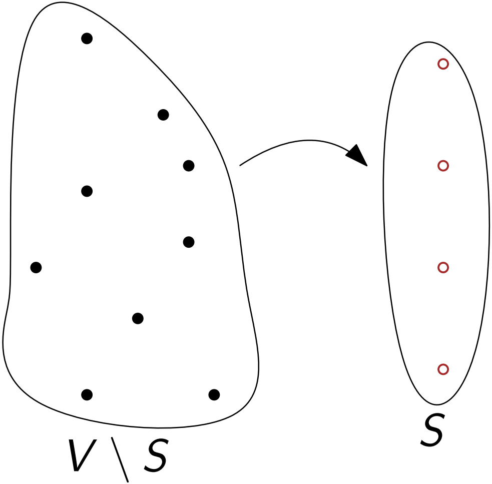
  if ??? **then**

    return S

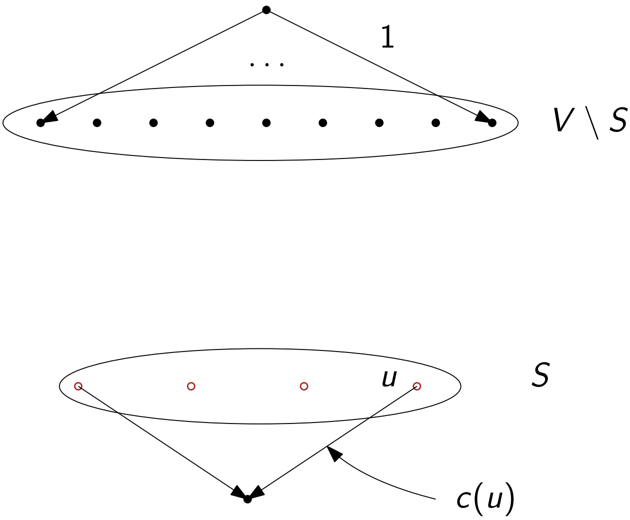
return :(

Solution verification

- Given $S \subseteq V$,
- $\exists$ assignment of $V \setminus S$ to S
- respecting capacities of S
- of cost $(1 + 2\varepsilon)\,\mathrm{OPT}$?



Solution verification: using network flows



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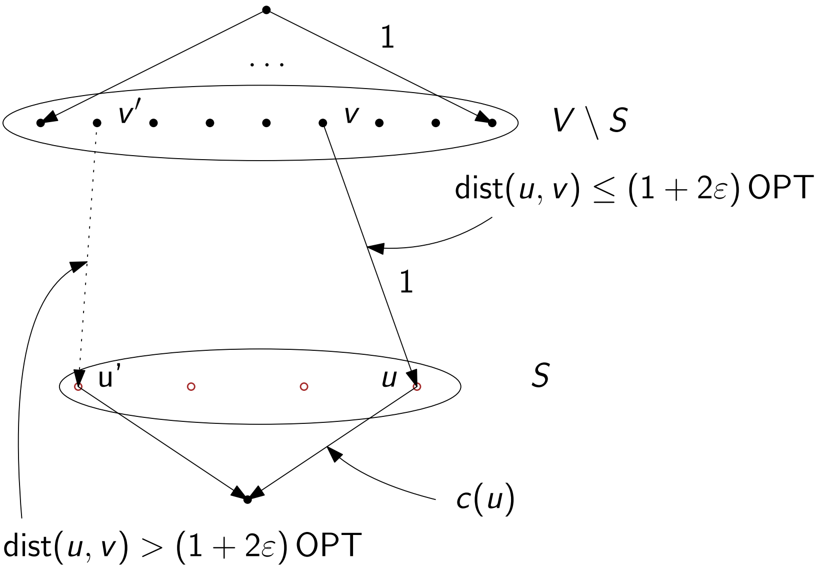

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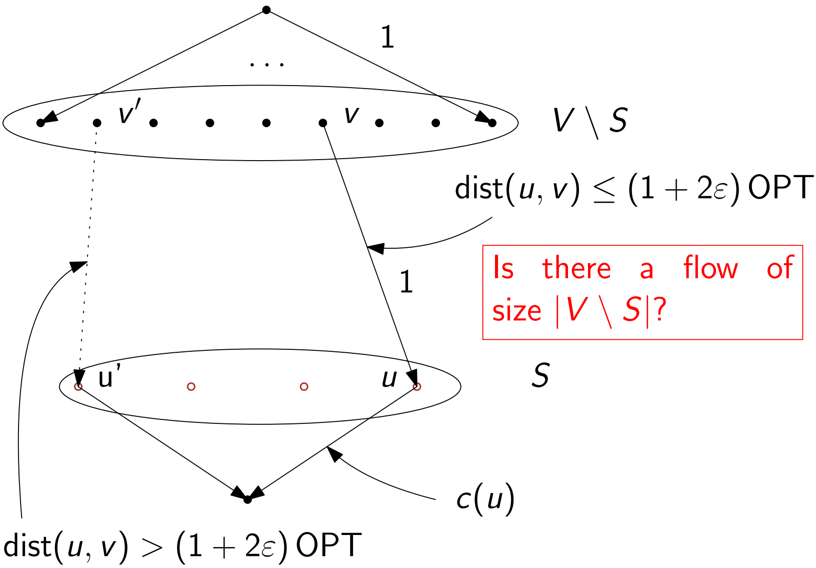

CKC algorithm

```
guess OPT
for all K ⊆_multiset Y with |K| ≤ k do
    S ← ∅
    for all v ∈ K do
        w ← vertex with the highest capacity in B_{ε OPT}(v) \ S
        S ← S ∪ {w}
    if ∃flow from V \ S to S of size |V \ S| then
        return assignment corresponding to that flow
return :(
```

Conclusion

	Doubling Dimension (Δ)	Highway dimension (h)
CAPACITATED k-CENTER	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Theorem 2	$\exists c > 1$: no c-approximation in $\mathcal{O}_\varepsilon\left(f(k,h) \cdot \text{poly}(n)\right)^{\dagger,\S}$ Theorem 1
k-CENTER	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Feldmann, Marx. 2020	$f(k,h,\varepsilon) \cdot \text{poly}(n)^{\dagger}$ Becker, Klein, Saulpic. 2018
k-MEDIAN, k-MEANS, FACILITY LOCATION	$2^{(1/\varepsilon)^{\mathcal{O}(\Delta^2)}} \cdot \text{poly}(n)$ Cohen-Addad, Feldmann, Saulpic. 2021	$n^{(2h/\varepsilon)^{\mathcal{O}(1)}}$ Feldmann, Saulpic. 2021
TSP, STEINER TREE	$\exp\{2^{\mathcal{O}(\Delta)} \cdot (4\Delta \log n/\varepsilon)^{\Delta}\}$ Talwar. 2004	$\exp\left\{\text{polylog}(n)^{\mathcal{O}(\log^2(h/\varepsilon))}\right\}$ Feldmann, Fung, Könemann, Post. 2018

$\dagger$: f: computable function

§: unless FPT = W[1]

Conclusion

	Doubling Dimension (Δ)	Highway dimension (h)
Capacitated k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Theorem 2	$\exists c > 1$: no c-approximation in $\mathcal{O}_\varepsilon(f(k,h) \cdot \text{poly}(n))^{\dagger,\S}$ Theorem 1
k-Center	$k^k/\varepsilon^{\mathcal{O}(k\Delta)} \cdot \text{poly}(n)$ Feldmann, Marx. 2020	$f(k,h,\varepsilon) \cdot \text{poly}(n)^\dagger$ Becker, Klein, Saulpic. 2018
k-Median, k-Means, Facility Location	$2^{(1/\varepsilon)^{\mathcal{O}(\Delta^2)}} \cdot \text{poly}(n)$ Cohen-Addad, Feldmann, Saulpic. 2021	$n^{(2h/\varepsilon)^{\mathcal{O}(1)}}$ Feldmann, Saulpic. 2021
TSP, Steiner Tree	$\exp\{2^{\mathcal{O}(\Delta)} \cdot (4\Delta \log n/\varepsilon)^\Delta\}$ Talwar. 2004	$\exp\left\{\text{polylog}(n)^{\mathcal{O}(\log^2(h/\varepsilon))}\right\}$ Feldmann, Fung, Könemann, Post. 2018

†: f: computable function   §: unless FPT = W[1]

Thank you for your attention!

Questions, comments, . . . ?